A Small Course on Indistinguishability Operators

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This is a short introduction to indistinguishability operators. Section 1 contains the definition of these operators and gives basic reasons for their study. Section 2 explains the two most popular ways to generate them, depending on the way the data are given. Section 3 introduces the idea of extensionality and shows that extensional fuzzy subsets with respect to a T-indistinguishability operator are the only fuzzy observable fuzzy subsets. Two maps that give upper and lower approximations of a fuzzy subset by extensional ones are studied. Perhaps the most interesting section is the last one (Section 4). It contains a number of comments and results that can be useful both to people interested in theoretical research and to people interested in applications.

For people not familiarized with t-norms, I recommend the excellent book by Klement, Mesiar, Pap ([33]) or if they are in a hurry to replace "t-norm" by "product" or "minimum".

More information and results on indistinguishability operators can be found in

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For any suggestion, comment or question, please do not hesitate to send me an e-mail (j.recasens@upc.edu).

1 Introduction

The notion of equality is essential in any theory since it allows to classify the objects it deals with.

Classifying is one of the most important processes in knowledge since it permits to relate, construct, generalize, find general laws, etc. It is unconceivable a scientific knowledge without an equality that allows to classify the objects it studies.

As a first approach to the concept of equality we can take Leibnitz's Law of Identity:

> Two objects are identical if and only if they have all their properties in common in a given universe of discourse.

Note the relativism of this law, since two objects can be equal in a universe and different in another one. But once fixed a universe consisting of a set of elements and properties that they satisfy completely or not at all, Leibnitz's Identity Law produces an equivalence relation on it.

In many real situations the objects do not necessarily satisfy (or not) a property categorically, but in general they satisfy it at some level or degree (think for example of the property "to be reach"). In these cases, properties are fuzzy concepts and the same happens with the Identity Law. We can not talk about identical objects, but a certain degree of similarity must be introduced. In this way, the equality turns to a fuzzy concept. Hence, a model of equality useful in different branches of knowledge must be based on the concept of indistinguishability, since in a theory two element are considered as equal if they are indistinguishables at a certain level. Classical (crisp) equivalence relations are to rigid to model this kind of equality and here appears the necessity of introducing fuzzy equivalence relations.

This considerations lead to the following central definition [44], [42].

Definition 1.1. *Let X be a universe and T a t-norm. A T-indistinguishability operator E on X is fuzzy relation $E : X \times X \to [0, 1]$ on X satisfying for all $x, y, z \in X$*

1. $E(x, x) = 1$ *(Reflexivity)*
2. $E(x, y) = E(y, x)$ *(Symmetry)*
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ *(T-Transitivity)*

$E(x, y)$ is interpreted as the degree of indistinguishability (or similarity) between x and y.

The three properties fuzzify the ones of a crisp equivalence relation. Reflexivity expresses the fact that every object is completely indistinguishable from itself. Symmetry says that the degree in which x is indistinguishable from y coincides with the degree in which y is indistinguishable from x. Transitivity deserves a special attention. As early as in 1901, H. Poincaré showed interest in this property. He states that in the physical word, *equal* actually means *indistinguishable*, since when we assert that two objects are equal, the only thing we can be sure of is that there is impossible to distinguish them. This consideration leads to the paradox that two objects A and B can be considered as equal, B can be equal (indistinguishable) to C but in turn A and B can be different (distinguishables); i.e.: the following situation can happen:

$$A = B \text{ and } B = C, \text{ but } A \neq C.$$

So Poincaré denies full transitivity in the real word.

T-transitivity in T-indistinguishability operators tries to overcome this paradox considering degrees of indistinguishability between objects. It gives a threshold to $E(x, z)$ knowing $E(x, y)$ and $E(y, z)$, since it is not reasonable in many cases to have three objects with high degree of indistinguishability between x and y and between y and z, but with x and z very distinguishables.

In a more logical context, transitivity expresses that the following proposition is true.

> If x is indistinguishable from y and y is indistinguishable from z, then x is indistinguishable from z.

In order to be useful in modeling different equalities, transitivity should be flexible. This can be achieved with the selection of a particular t-norm.

In this sense, it is worth recalling that when we use the product t-norm, we obtain the so-called possibility relations introduced and studied by K. Menger in probabilistic metric spaces. If we choose the Łukasiewicz t-norm, we obtain the relations called likeness introduced by E. Ruspini. For the minimum t-norm we obtain similarity relations [44].

Similarity relations are widely used in Taxonomy since they are tightly related to hierarchical trees. Indeed, given a similarity relation E on X and $\alpha \in [0,1]$, the crisp relation $\sim_\alpha$ on X defined by $x \sim_\alpha y$ if and only if $Ex, y) \geq \alpha$ (the α-cut of E) is a crisp equivalence relation and generates a partition of X.

2 Generation of T-indistinguishability operators

There are several ways to generate T-indistinguishability operators depending on the way the data are given and on the use we want to make of it. Here we will only present the construction of the T-transitive closure of a reflexive and symmetric fuzzy relation and the generation of T-indistinguishability operators via the Representation Theorem, though other methods like the construction of decomposable indistinguishability operators and the calculation of openings of a given reflexive and symmetric fuzzy relation R (i.e.: maximal T-indistinguishability operators among the ones smaller or equal than R) are also of interest.

2.1 sup-T product

Definition 2.1. *Let R and S be two fuzzy relations on X and T a continuous t-norm. The $\sup -T$ product of R and S is the fuzzy relation $R \circ S$ on X defined for all $x, y \in X$ by*

$$(R \circ S)(x, y) = \sup_{z \in X} T(R(x, z), S(z, y)).$$

Since the $\sup -T$ product is associative for continuous t-norms, we can define for $n \in N$ the n^{th} power R_T^n of a fuzzy relation R:

$$R_T^n = \overbrace{R \circ \ldots \circ R}^{n\ times}.$$

Definition 2.2. *Let R be a fuzzy relation on a set X and T a continuous t-norm. The transitive closure of R with respect to T is the fuzzy relation*

$$R^T = \sup_{n \in N} R_T^n.$$

Proposition 2.3. *Let R be a reflexive and symmetric fuzzy relation on a finite set X of cardinality n. Then*

$$R^T = R_T^{n-1}.$$

Proposition 2.4. *Let R be a reflexive and symmetric fuzzy relation on X and T a continuous t-norm. The T-transitive closure $\overline{R}$ of R is the smallest T-indistinguishability operator on X satisfying $R \leq \overline{R}$.*

So the transitive closure is a way to get an upper approximation of a given reflexive and symmetric fuzzy relation by a T-indistinguishability operator.

If the cardinality n of X is finite, we can represent a fuzzy relation R on X by a square $n \times n$ matrix. The matrix is symmetric if and only if R is; R is reflexive if and only if the diagonal of the matrix consists of ones.

Example 2.5. *Let R be the fuzzy relation given by the matrix*

$$\begin{pmatrix} 1 & 0.9 & 0.3 & 0.4 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.3 & 0.5 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix}.$$

The transitive closures with respect to the t-norms of Łukasiewicz, product and minimum respectively are

$$\begin{pmatrix} 1 & 0.9 & 0.4 & 0.4 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.4 & 0.5 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9 & 0.45 & 0.4 \\ 0.9 & 1 & 0.5 & 0.45 \\ 0.45 & 0.5 & 1 & 0.9 \\ 0.4 & 0.45 & 0.9 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9 & 0.5 & 0.5 \\ 0.9 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.9 \\ 0.5 & 0.5 & 0.9 & 1 \end{pmatrix}.$$

2.2 Representation Theorem

The Representation Theorem allows us to generate a T-indistinguishability operator on a set X from a family of fuzzy subsets of X, and reciprocally states that every T-indistinguishability operator can be obtained in this form.

Let us first recall the concept of residuation of a t-norm.

Definition 2.6. *The residuation $\overrightarrow{T}$ of a t-norm T is defined by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y).$$

Definition 2.7. *The biresiduation $\overleftrightarrow{T}$ of a t-norm T is defined by*

$$\overleftrightarrow{T}(x,y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)) = \min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

Example 2.8.

1. *If T is an Archimedean t-norm with additive generator t, then $\overleftrightarrow{T}(x,y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0,1]$.*

   As special cases,

   - *If T is the Łukasiewicz t-norm, then $\overleftrightarrow{T}(x,y) = 1 - |x-y|$ for all $x, y \in [0,1]$.*
   - *If T is the product t-norm, then $\overleftrightarrow{T}(x,y) = \min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.*

2. *If T is the minimum t-norm, then* $\overleftrightarrow{T}(x,y) = \begin{cases} \min(x,y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

Lemma 2.9. *Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x,y) = \overleftrightarrow{T}(\mu(x), \mu(y))$$

is a T-indistinguishability operator.

in the crisp case, when $\mu = A$ is a crisp subset of X, E_A generates a partition of X into A and its complementary $X - A$, since in this case $E_A(x,y) = 1$ if and only if x and y both belong to A or to $X - A$.

Theorem 2.10. *Representation Theorem [43]. Let R be a fuzzy relation on a set X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = \inf_{i\in I} E_{\mu_i}(x,y).$$

$(\mu_i)_{i\in I}$ is called a generating family of R.

The Representation Theorem provides us with a method to generate a T-indistinguishability operator from fuzzy subsets.

In particular, given a reflexive and symmetric fuzzy relation R on X, we can build the T-indistinguishability operator $\underline{R}$ generated by the set of the columns of R (i.e. the fuzzy subsets $R(x,\cdot)$, $x \in X$).

Proposition 2.11. $\underline{R} \le R$.

Example 2.12. *Considering the same fuzzy relation R of Example 2.5 given by the matrix*

$$\begin{pmatrix} 1 & 0.9 & 0.3 & 0.4 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.3 & 0.5 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix},$$

the obtained indistinguishabilities with respect to the t-norms of Łukasiewicz, product and minimum respectively are

$$\begin{pmatrix} 1 & 0.8 & 0.3 & 0.4 \\ 0.8 & 1 & 0.4 & 0.4 \\ 0.3 & 0.4 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.6 & 0.3 & 0.\widehat{3} \\ 0.6 & 1 & 0.\widehat{3} & 0.4 \\ 0.3 & 0.\widehat{3} & 1 & 0.75 \\ 0.\widehat{3} & 0.4 & 0.75 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 & 0.4 \\ 0.3 & 0.3 & 1 & 0.3 \\ 0.3 & 0.4 & 0.3 & 1 \end{pmatrix}.$$

3 Extensional sets

Extensional fuzzy subsets with respect to a T-indistinguishability operator E play a central role since they are the only observable sets taking E into account. In the crisp case, when E is a crisp equivalence relation on a universe X, the only crisp subsets from which something can be said if E is considered are only the union of equivalence classes of E (and intersections if we want to add the empty set). The equivalence classes give the granularity in X. If E is a fuzzy relation, extensional sets play this role and they show the granularity generated by E.

Again in the crisp case, a given subset A of X can be approximated from above and from below by observable (union of equivalence classes) subsets of X, this being the core of the theory of rough sets. Similarly, for a fuzzy subset μ upper and lower approximations of μ by extensional fuzzy subsets can be obtained introducing two operators ϕ_E and ψ_E.

Definition 3.1. *Let E be a T-indistinguishability operator on a set X and μ a fuzzy subset of X. μ is extensional with respect to E if and only if for all $x, y \in X$*

$$T(E(x,y), \mu(x)) \leq \mu(y).$$

H_E will denote the set of all extensional fuzzy subsets with respect to E.

Extensional fuzzy subsets are also called generators since they belong to a generating family of E in the sense of the Representation Theorem 2.10.

Proposition 3.2. *Let E be a T-indistinguishability operator on a set X and μ a fuzzy subset of X. μ is extensional with respect to E if and only if $E_\mu \geq E$.*

There is a nice characterization of the sets of fuzzy subsets of a universe X that are exactly the extensional subsets with respect to a T-indistinguishability operator on X. Let $F(X)$ be the set of all fuzzy subsets of X.

Proposition 3.3. *[9] Given a set H of fuzzy subsets of X, there exists a T-indistinguishability operator E on X such that $H = H_E$ if and only if for all fuzzy subsets μ of X and for all $\alpha \in [0,1]$,*

1. $T(\alpha, \mu) \in H$
2. $\overrightarrow{T}(\alpha|\mu) \in H$
3. $\overrightarrow{T}(\mu|\alpha) \in H$
4. $(H, \leq)$ *is a complete sublattice of* $(F(X), \leq)$.

Definition 3.4. *Let E be a T-indistinguishability operator on X. The map $\phi_E : F(X) \to F(X)$ is defined by*

$$\phi_E(\mu)(x) = \sup_{y \in X}\{\mu(y), E(x,y)\}, \ \text{for all } x \in X.$$

Some properties of ϕ_E.

Proposition 3.5.

1. $\phi_E(\mu) \geq \mu$
2. *If* $\mu \leq \nu$, *then* $\phi_E(\mu) \leq \phi_E(\nu)$
3. $\phi_E(\mu \vee \nu) = \phi_E(\mu) \vee \phi_E(\nu)$.

ϕ_E sends each fuzzy subset μ to the smallest extensional fuzzy subset that contains it. In particular,

Proposition 3.6. $\phi_E(\mu) = \mu$ *if and only if* $\mu \in H_E$.

Corollary 3.7.

1. $\phi_E(\mu) = \wedge_{\nu \in H_E}\{\nu \geq \mu\}$
2. $\phi_E^2 = \phi_E$
3. $\phi_E(\mu_\alpha) = \mu_\alpha$ *for each constant fuzzy subset* μ_α.

In particular, ϕ_E is a fuzzy closure operator.

Proposition 3.8.

1. $\phi_E(\{x\} = E(x, \cdot)$
2. $\phi_E(A) = \vee_{x \in A} E(x, \cdot)$.

The fuzzy subsets $E(x, \cdot)$ are called the columns of E. Last propositions states that a singleton is fuzzified to its corresponding column when we take E into account and that the column is the best upper approximation of its corresponding singleton.

In this sense, H_E can be interpreted as the set of fuzzy subsets of the quotient set X/E (i.e. $H_E = F(X/E)$) and $\phi_E : F(X) \to F(X/E)$ is the canonical map.

Let us turn our attention to the map ψ_E that sends each fuzzy subset to the greatest extensional fuzzy subset contained in it.

Definition 3.9. *Let E be a T-indistinguishability operator on X. The map $\psi_E : F(X) \to F(X)$ is defined by*

$$\psi_E(\mu)(x) = \inf_{y \in X} \{\overrightarrow{T} (E(x, y)|\mu(y)\} \ for\ all\ x \in X.$$

Properties of ψ_E are dual to those of ϕ_E.

Proposition 3.10.

1. $\psi_E(\mu) \leq \mu$
2. *If* $\mu \leq \nu$, *then* $\psi_E(\mu) \leq \psi_E(\nu)$
3. $\psi_E(\mu \wedge nu) = \psi_E(\mu) \wedge \psi_E(\nu)$
4. $\psi_E(\mu) \in H_E$
5. $\psi_E(\mu) = \vee_{\nu \in H_E} \{\nu \leq \mu\}$
6. $\psi_E^2 = \psi_E$
7. $\psi_E(\mu_\alpha) = \mu_\alpha$ *for all constant* μ_α.

In particular, ψ_E is a fuzzy interior operator.

4 Comments

This section is a medley of comments and results that intend to be useful both to people working in theoretical aspects of fuzzy logic and to people interested in their applications.

Fuzzy rough sets

If $\sim$ is a crisp relation of X, then a crisp subset $A \subseteq X$ can be approximated from bellow by the union of all the equivalence classes of $\sim$ contained in A and from above by the union of all the equivalence classes with non-empty intersection with A. These two approximations of A by "observable" sets is called a rough set, a concept introduced by Pawlak [35]. In the fuzzy case, this upper and lower approximations are obtained by the operators ϕ_E and ψ_E. (see [34], [5]).

Rough set approximations have been interpreted as the necessity and possibility in modal logic. A similar interpretation can be made of the operators ϕ_E and ψ_E.

Triangular and trapezoidal fuzzy numbers. Fuzzification

An indistinguishability operator provides granularity to the universe. Proposition 3.8 finds the smallest extensional fuzzy subset $\phi_E(A)$ containing a given crisp set A. If the universe X is part of the real line and E is defined by $E(x, y) = \max(1 - \alpha|x - y|, 0)$ for some $\alpha > 0$ (E is a T-indistinguishability operator for the Łukasiewicz t-norm), the fuzzification of a point and an interval are a triangular and a trapezoidal number respectively. This gives a sound theoretical interpretation to the usual fuzzification of points and intervals by these numbers: They are the less specific extensional fuzzy subsets containing the point or interval assuming the existence of the indistinguishability operator E [28].

Indistinguishability operators and metric spaces

Recall that a map $m : X \times X \to \mathbb{R}$ is a pseudometric and (X, m) a pseudometric space if and only if

1. $m(x, x) = 0$

2. $m(x, y) = m(y, x)$

3. $m(x, y) + m(y, z) \geq m(x, z)$ (Triangular inequality)

If $m(x, y) = 0$ implies $x = y$, then it is a metric.

There are two connexions between indistinguishability operators and pseudometric spaces [43]. The first one is that if T is greater or equal than the Łukasiewicz t-norm, then a fuzzy relation E on X is a T-indistinguishability operator if and only if $m = 1 - E$ is a pseudometric on X. $1 - E$ is a metric if and only if E separates points (i.e. $E(x, y) = 1$ implies $x = y$).

In fact, for any continuous t-norm T and φ a strong negation [41], if E is a T-indistinguishability operator, then $\varphi \circ E$ is a so-called S-metric [43] where S is the dual t-conorm of T with respect to φ ($S(x, y) = \varphi \circ T(\varphi \times \varphi)$).

The second relation between indistinguishability operators and pseudometric spaces is that if T is a continuous Archimedean t-norm and t an additive generator of T, then

1. If E is a T-indistinguishability operator on X, then $t \circ E$ is a pseudometric on X. It is a metric if and only of E separates points.

2. If m is a pseudometric on X, then $t^{[-1]} \circ m$ is a T indistinguishability operator. E separates points if and only if m is a metric.

   Actually, (X, E) is a kind of Generalized metric space [40] and the results of these spaces can be applied to them. A study of isometries between universes with indistinguishability operators can be found in [27], [24].

Similarities and ultrametrics

A pseudo metric $m : X \times X \to \mathbb{R}$ is a pseudo ultrametric if the triangular inequality is strengthen by

$$\max(m(x, y), m(y, z)) \geq m(x, z).$$

Ultrametric spaces have a very special behaviour.

Proposition 4.1.

1. *If $B_r(x)$ denotes the ball of centre x and radius r, and $y \in B_r(x)$, then $B_r(x) = B_r(y)$. (All elements of a ball are its centre).*

2. *If two balls have non-empty intersection, then one of them is contained in the other one.*

Proposition 4.2. *Let E be a fuzzy relation on a set X. E is a similarity (a T-indistinguishability operator for the minimum t-norm) if and only if $1-E$ is a pseudo ultrametric.*

Due to this proposition, interesting results on similarities follow. One of them is that the cardinality of $\mathrm{Im}E = \{E(x,y)\}$ is smaller or equal than the cardinality of X. In particular, if X is finite of cardinality n and E is identified with a matrix, then the number of different entries of the matrix is less or equal than n, which simplifies calculations and storage.

Another important result has already been stated in the introductory section. Recall that for a fuzzy relation on X and $\alpha \in [0,1]$, the α-cut of E is the set E_α of pairs $(x,y) \in X \times X$ such that $E(x,y) \geq \alpha$. If $\alpha \geq \beta$, then $E_\alpha \leq E_\beta$.

Proposition 4.3. *Let E be a fuzzy relation on X. E is a similarity relation on X if and only if for each $\alpha \in [0,1]$, the α-cut of E is a crisp equivalence relation.*

So every similarity on a finite set X generates a hierarchical tree on X and reciprocally, every indexed hierarchical tree generates a similarity on X.

If R is a reflexive and symmetric fuzzy relation on X, then its transitive closure with respect to the minimum is a similarity E. The partitions of the hierarchical tree generated by E coincide with the ones obtained from R by single linkage.

Partitions

A family of fuzzy subsets of X generates a T-indistinguishability operator E by the Representation Theorem 2.10. The following result states when the fuzzy subsets of the family are columns of E [29] [30].

Proposition 4.4. *Let $(\mu_i)_{i\in I}$ be a family of normal fuzzy subsets of X (a fuzzy subset μ is normal if and only if there exists $x \in X$ with $\mu(x) = 1$) and $(x_i)_{i\in I}$ a family of elements of X with $\mu_i(x_i) = 1$. The following two statements are equivalent.*

1. *There exists a T-indistinguishability operator on X such that $\phi_E(x_i) = \mu_i$) for all $i \in I$, i.e. $\mu_i = E(x_i, \cdot)$.*

2. *For all* $i, j \in I$ $\sup_{x\in X} T\,(\mu_i(x), \mu_j(x)) \leq \inf_{x\in X} \overset{\leftrightarrow}{T}\,(\mu_i(x), \mu_j(x))$.

Note that if $T(\mu_i(x), \mu_j(x)) = 0$ for every $x \in X$ and $i \neq j$, the second statement of the proposition holds trivially.

Given a t-norm T and a dual t-conorm S with respect to a strong negation, an S, T-fuzzy partition is defined as follows.

Definition 4.5. *An* S, T*-fuzzy partition of universe* X *is a (finite) family* P *of fuzzy subsets of* X $P = \{\mu_1, \mu_2, ..., \mu_n\}$ *such that*

1. $T(\mu_i(x), \mu_j(x)) = 0$ *for every* $x \in X$ *and* $i \neq j$.

2. $S(\mu_1(x), \mu_2(x), ..., \mu_n(x)) = 1$ *for all* $x \in X$.

Corollary 4.6. *Let* $P = \{\mu_1, \mu_2, ..., \mu_n\}$ *be a* S, T*-fuzzy partition of* X *with all* μ_i *normal fuzzy sets and* E *the* T*-indistinguishability operators generated by the family* P*. The elements of* P *are columns of* E*.*

Fuzzy points

The granularity generated by a crisp equivalence relation is given by its equivalence classes. In the fuzzy case, there are two ways to generalize the granularity. One is taking as granules the fuzzy equivalence classes of the indistinguishability operator, i.e. its columns. The other one is introducing the concept of fuzzy point.

Definition 4.7. *Let* E *be a* T*-indistinguishability operator on* X*. A fuzzy subset* μ *of* X *is a fuzzy point if and only if*

1. μ *is extensional with respect to* E

2. $T(\mu(x), \mu(y)) \leq E(x, y)$ *for all* $x, y \in X$.

In particular, the columns are the only normal fuzzy points of E.

Fuzzy points have been studied in [30], [7] and in [26] the existence of maximal fuzzy points has been established.

Dimension and basis

Going back th the Representation theorem 2.10, different families of fuzzy subsets can generate the same T-indistinguishability operator E. This gives great interest to the theorem, since if we interpret the elements of the family as degrees of matching between the elements of X and a set of prototypes, we can use different features, giving different interpretations to E. Among the generating families of a relation, the ones with low cardinality are of special interest, since they have an easy semantical interpretation and also because the information contained in the matrix can be packed in a few fuzzy subsets.

Definition 4.8. *Let E be a T-indistinguishability operator on X. The dimension of E is the minimum of the cardinalities of the generating families of E in the sense of the Representation Theorem. A generating family with this cardinality is called a basis of E.*

In [4],[18] a couple of algorithms to find the dimension and a basis of a similarity is given. A geometric approach and an algorithm for calculating the dimension and a basis of T-indistinguishability operators with T Archimedean can be found in [4],[21],[22].

One-dimensional indistinguishability operators

Among the indistinguishability operators, one-dimensional ones play a special role for two main reasons. They appear naturally when we generate a T-indistinguishability operator from a fuzzy subset and they have a simpler structure that higher dimensional ones.

The one-dimensional similarities are very easy to be detected (Proposition 4.10).

Definition 4.9. *If E is a T-indistinguishability operator on X, let $\sim$ be the crisp relation on X $x \sim y$ if and only of $E(x,y) = 1$. The fuzzy relation $\overline{E}$ on $X/\sim$ defined by $\overline{E}(\overline{x}, \overline{y}) = E(x,y)$ is a T-indistinguishability operator that separates points.*

Proposition 4.10. *[18] Let E be a similarity on X. E is one dimensional if and only if there exists a column of $\overline{E}$ with all its elements different from each other.*

For Archimedean t-norms, the dimension of a T-indistinguishability relation is tightly related to the betweenness relation it defines (Proposition 4.13).

Definition 4.11. *[32] A ternary relation $B \subseteq X \times X \times X$ is a betweenness relation on X if and only if for all $(x,y,z) \in B$*

1. $x \neq y \neq z \neq x$

2. $(z, y, x) \in B$

3. $(y, z, x) \notin B$, $(z, x, y) \notin B$

4. *If moreover* $(x, z, t) \in B$, *then* $(x, y, t) \in B$ *and* $(y, z, t) \in B$.

If $(x, y, z) \in X$, *then we say that* y *is between* x *and* z. *If for any three different elements of* X *one is between the other two, the betweenness relation is called total or linear.*

Example 4.12.

1. *If* (X, m) *is a metric space, then the relation* $(x, y, z) \in B$ *if and only if* $m(x, y) + m(y, z) = m(x, z)$ *is a betweenness relation on* X.

2. *If* E *is a* T*-indistinguishability operator separating points with* T *Archimedean and* $E(x, y) \neq 0$, *then the relation* $(x, y, z) \in B$ *if and only if* $T(E(x, y), E(y, z) = E(x, z)$ *is a betweenness relation on* X.

Proposition 4.13. *A* T*-indistinguishability operator on a set* X *with* T *Archimedean and* $E(x, y) \neq 0$ *is one-dimensional if and only if the betweenness relation generated by* $\overline{E}$ *on* $\overline{X}$ *is linear.*

The exact relation between the dimension of a T-indistinguishability operator E and the cardinality of the betweenness relation it generates is of combinatorial nature and seems very difficult to obtain [38].

Approximating indistinguishability operators

In some cases it can happen that a T-indistinguishability operator has a high dimension but slight changes of the entries of the matrix provide a close matrix to the original one of lower dimension. How to approximate a T-indistinguishability operator T Archimedean by a one dimensional one inspired in Saaty's preference matrices can be found in [20].

Fuzzy topologies

As we have seen in Section 3, ϕ_E is a fuzzy closure operator and therefore H_E are closed fuzzy sets of a fuzzy topology as defined in [31] and ψ_E is a fuzzy interior operator, hence defining a fuzzy topology on H_E.

On the other hand, indistinguishability operators generate (crisp) topologies in a rather natural way.

These crisp and fuzzy topologies and their relation have been studied in [4],[21],[19].

Aggregation of indistinguishability operators

In many situations, there can be more than one indistinguishability operator defined on a universe. Let us suppose, for example that we have a set of instances defined by some features. We can generate an indistinguishability operator from each feature. Also we can have some prototypes and again we can define an operator from each of them in our universe. In these cases we may need to aggregate the obtained relations. The usual way to do it is calculating the minimum (or infimum) of them which is an indistinguishability operator thanks to the Representation Theorem 2.10. Although this has a very clear interpretation in fuzzy logic since the Infimum is used to model the universal fuzzy quantifier $\forall$, it leads many times to undesirable results in applications. The reason is that the minimum has a drastic effect. If for example two objects of our universe are very similar or indistinguishable for all but one indistinguishability operator, but for this particular one very different, then the result applying the minimum will give this last measure and forget all the other ones. This can be reasonable and useful if we need a perfect matching with respect to all our relations, but this is not the case in many situations. When we need to take all the relations into account in a less dramatic way, we need to use other ways to aggregate them. Since if R and S are T-transitive fuzzy relations with respect to a t-norm T then $T(R,S)$ is also a T-transitive fuzzy relation, it seems at first glance that this could be a good way to aggregate them. Nevertheless, if we aggregate in this way we obtain relations with very low values; in the case of non-strict Archimedean t-norms this is even worse, since many times we will get almost all values of the obtained relation equal to zero. Therefore, other ways have to be found.

Definition 4.14. *[1] Let $t : [0,1] \to [-\infty,+\infty]$ be a monotonic map and $x_1, x_2, ..., x_n \in [0,1]$. The quasi-arithmetic mean of $x_1, x_2, ..., x_n$ generated by t is*

$$m_t(x_1, x_2, ..., x_n) = t^{-1}\left(\frac{t(x_1) + t(x_2) + ... + t(x_n)}{n}\right).$$

Example 4.15.

1. *If $t(x) = 1 - x$, then m_t is the arithmetic mean.*

2. *If $t(x) = -\log x$, then m_t is the geometric mean.*

If T is an Archimedean t-norm with additive generator t, we can consider the quasi-arithmetic mean generated by t. Then we have the following result.

Proposition 4.16. *Let $E_1, E_2, ..., E_n$ be T indistinguishability operators on X with T an Archimedean t-norm with additive generator t. Then $m_t(E_1, E_2, ..., E_n)$ is a T-indistinguishability operator.*

Another way to aggregate indistinguishability operators is using OWA operators.

Proposition 4.17. *Let $E_1, E_2, ..., E_n$ be T indistinguishability operators on X with T an Archimedean t-norm with additive generator t. Let $s : \mathbb{R}^n \to \mathbb{R}$ be the OWA operator with weights $(p_1, p_2, ..., p_n)$ ($s(x_1, x_2, ..., x_n) = \sum_{i=1}^n p_i x_{(i)}$ where $x_{(i)}$ is the i^{th} largest of the x_i) and t a generator of T. The fuzzy relation E on X defined by*

$$E(x, y) = t^{[-1]}\left(s\left(t(E_{(1)}(x, y)) + t(E_{(2)}(x, y)) + ... + t(E_{(n)}(x, y))\right)\right)$$

is a T-indistinguishability operator if and only if $p_i \geq p_j$ for $i < j$.

In some cases we have to aggregate a non-finite number of relations, for example if we need to compare two fuzzy sets of the real line. Let us suppose that we have a family of T-indistinguishability operators $(E_i)_{i\in[a,b]}$ on a set X with the indices in the interval $[a, b]$ of the real line and that for every couple (x, y) of X the map $f_{(x,y)} : [a, b] \to R$ defined by $f_{(x,y)}(i) = E_i(x, y)$ is integrable in some sense.

Definition 4.18. *The aggregation of the family $(E_i)_{i\in[a,b]}$ with respect to T is the T-indistinguishability operator E on X defined for all $x, y \in X$ by*

$$E(x, y) = t^{[-1]}(\frac{1}{b-a}\int_a^b t(E_i(x, y))di)$$

More information on the aggregation of T-indistinguishability operators is available in [25],[36], [37].

Observational entropy. Fuzzy observational decision trees

Shannon's measure of entropy is defined by

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

This measure was thought within the frame of communication theory, specifically for facing issues concerning channel reliability and reduction of transmission cost, but ignoring the semantic content of the messages involved. Let us suppose the existence of a T-indistinguishability operator on X. Then the occurrence of two different events, but indistinguishable by the indistinguishability relation defined, will count as the occurrence of the same event when measuring the "observational" entropy defined as follows.

Definition 4.19. *Let E a T-indistinguishability operator on a set X. The observation degree of $x_j \in X$ is defined by:*

$$\pi(x_j) = \sum_{x \in X} p(x) E(x, x_j).$$

This definition has a clear interpretation: the possibility of observing x_j is given by the probability that x_j really happens plus the probability of occurrence of some element "very similar" to x_j, weighted by the similarity degree.

Definition 4.20. *The quantity of information received by observing x_j is defined by:*

$$C(x_j) = -\log_2 \pi(x_j).$$

Definition 4.21. *Given a T-indistinguishability operator E on X, and P a probability distribution on X, the observational entropy (HO) of the pair (E, P) is defined by:*

$$HO(E, P) = \sum_{x \in X} p(x) C(x).$$

This measure has been studied in [16] and has been applied to the generation of fuzzy decision trees when there is an indistinguishability operator defined on the universe of discourse [15].

Fuzzy maps

Fuzzy maps fuzzify the concept of map between two universes. They have been used in different fields like vague algebras [12] and fuzzy numbers [28] and have been proved useful in order to understand fuzzy reasoning, especially reasoning based on fuzzy rules [30].

Definition 4.22. *Let E and F be two T-indistinguishability operators on X and Y respectively. A fuzzy subset $R : X \times Y \to [0, 1]$ of $X \times Y$ is a fuzzy map on $X \times Y$ (or from X to Y) if and only if for all $x, x' \in X$ and $y, y' \in Y$*

1. $T(E(x, x'), F(y, y'), R(x, y)) \leq R(x', y'))$
2. $T(R(x, y), R(x, y')) \leq F(y, y'))$.

Definition 4.23. *Let E, F be two T-indistinguishability operators on X and Y respectively. A map $f : X \to Y$ is extensional with respect to E and F if and only if*

$$E(x, y) \leq F(f(x), f(y)).$$

Extensional maps have been widely used in areas such as Approximate Reasoning [3], vague algebras [12], logic [14] among others.

Definition 4.24. *A fuzzy map R on $X \times Y$ is perfect if and only for any $x \in X$ there exists a unique $y_x \in Y$ with $R(x, y_x) = 1$.*

A perfect fuzzy map defines a (crisp) extensional map $f_R : X \to Y$: $f_R(x) = y_x$.

Reciprocally,

Proposition 4.25. *An extensional map $f : X \to Y$ defines a perfect fuzzy map R_f on $X \times Y$ by $R_f(x, y) = F(f(x), y)$.*

Indistinguishability operators and approximate reasoning

Fuzzy maps have been used by Klawonn in an excellent article [30] to interpret Mamdani's controllers in their context.

Consider a family of fuzzy rules of the form if ξ is A_i, then ν is B_i ($i \in I$) where the linguistic terms A_i and B_i are modelled by the fuzzy sets μ_{A_i} of X and ν_{B_i} of Y.

Depending on the way the entailment is modelled, a fuzzy relation ρ_L or ρ_U is generated by

$$\rho_U(x, y) = \wedge_{i\in I} \overrightarrow{T} (\mu_i(x)|\nu_i(y))$$

or

$$\rho_L(x, y) = \vee_{i\in I} T(\mu_i(x), \nu_i(y)).$$

Let us now consider a control problem where we want to find a map $f : X \to Y$ and we only know some values of f. The following result gives upper and lower bounds to f.

Proposition 4.26. *Let E and F be two T-indistinguishability operators on X and Y respectively and $f : X \to Y$ an (unknown) extensional map with respect to E and F. Let $A = \{x_1, x_2, ..., x_n\}$ be some elements of X and let f_n denote restriction of f to A (which is known). Let $\mu_i = \phi_E(x_i)$ and $\mu_i = \phi_F(f(x_i))$. Then*

$$\rho_L = R_{f_n} \le R_f \le \rho_U.$$

Vague algebras

Demirci has introduced the concept of vague algebra that basically consists of operations compatible with given indistinguishability operators [12].

As an example, let us see how a vague group is defined.

Definition 4.27. *Let E and $E_{X\times X}$ be T-indistinguishability operators on X and $X \times X$ respectively. A vague binary operation on X is a perfect fuzzy map $\tilde{\circ}$ from $X \times X$ to X.*

$\tilde{\circ}(x, y, z)$ means the degree in which z is $x \circ y$.

Definition 4.28. *A vague group $(X, \tilde{\circ})$ is a set X with a T-indistinguishability operator and a vague binary operation $\tilde{\circ}$ satisfying*

1. *$\forall a, b, c, d, m, q, w \in X, (T((\tilde{\circ}(b, c, d), \tilde{\circ}(a, d, m), \tilde{\circ}(a, b, q), \tilde{\circ}(q, c, w))) \le E_X(m, w))$ (Associativity).*
2. *There exists a (two sided) identity element $e \in X$ such that $\tilde{\circ}(e, a, a) = \tilde{\circ}(a, e, a) = 1$ for each $a \in X$.*

3. For each $a \in X$, there exists a (two-sided) inverse element $a^{-1} \in X$ such that $\tilde{\circ}(a^{-1}, a, e) = \tilde{\circ}(a, a^{-1}, e) = 1$.

The relation between vague groups and fuzzy groups and their application to fuzzy arithmetic has been studied in [13]

References

[1] J. Aczél. Lectures on functional equations and their applications. Academic Press, NY–London, 1966.

[2] Belohlávek, R. *Fuzzy Relational Systems: Foundations and Principles.* (2002) Kluwer Academic Publishers, New York.

[3] Boixader, D. T-indistinguishability operators and approximate reasoning via CRI. In: Dubois, D., Klement, E.P., Prade, H. (eds). Fuzzy Sets, Logics and Reasoning about Knowledge. 1 ed. Dordrecht: Kluwer Academic Publishers (1999) 255-268

[4] Boixader, D., Jacas, J., Recasens, J. Fuzzy equivalence relations: advanced material. Dubois, Prade Eds Fundamentals of Fuzzy Sets. Kluwer 1999. (261-290).

[5] Boixader, D., Jacas, J., Recasens, J. Upper and lower approximation of fuzzy sets. Int. J. of General Systems 29 (2000) 555-568.

[6] Boixader, D., Jacas, J., Recasens, J. Searching for meaning in defuzzification. Int. J. Uncertainty, Fuzziness and Knowledge-based Systems 7 (1999) 475-482.

[7] Boixader, D., Jacas, J., Recasens, J. A characterization of the columns of a T-indistinguishability operator. Proceedings FUZZ'IEEE-98 Congress. Anchorage (1998) 808-812.

[8] T. Calvo, A. Kolesárova, M. Komorníková, R. Mesiar (2002). Aggregation Operators: Properties, Classes and Construction Methods. In Mesiar, Calvo, Mayor Eds. *Aggregation Operators: New Trends and Applications.* Studies in Fuzziness and Soft Computing. Springer, 3-104.

[9] J. L. Castro, F. Klawonn. Similarity in Fuzzy Reasoning. Mathware & Soft Computing (1996).

[10] De Baets, B. and Mesiar, R., T-partitions. Fuzzy Sets and Systems 97 (1998) 211-223.

[11] De Baets, B. and Mesiar, R., Metrics and T-equalities. J. Math. Anal. Appl. (2002) 267 531-547.

[12] Demirci, M. Fundamentals of M-vague algebra and M-vague arithmetic operations. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 10 (2002) 25-75.

[13] Demirci, M., Recasens, J. Fuzzy Groups, Fuzzy Functions and Fuzzy Equivalence Relations. Fuzzy Sets and Systems 144 (2004) 441-458.

[14] Hajek P. *Metamathematics of fuzzy logic*. Kluwer (1998).

[15] Hernández, E., Recasens, J. A General Framework for Induction of Decision Trees under Uncertainty. Lawry, J., Shanahan, J., Ralescu, A. Eds. Modelling with Words. LNAI 2873. Springer (2003). 26-43.

[16] Hernández, E., Recasens, J. A reformulation of entropy in the presence of indistinguishability operators. Fuzzy Sets and Systems 128. (2002) 185-196.

[17] Jacas, J. On the generators of T-indistinguishability operators. Stochastica 12 (1988) 49-63.

[18] Jacas, J. Similarity Relations - The Calculation of Minimal Generating Families. Fuzzy Sets and Systems 35 (1990) 151-162.

[19] Jacas, J. Fuzzy Topologies induced by S-metrics. The Journal of fuzzy mathematics 1 (1993) 173-191.

[20] Jacas, J., Recasens, J. On some kinds of probabilistic relations. B.Bouchon-Meunier, L.Valverde, R.R.Yager Eds. Intelligent Systems with uncertainty. Elsevier. 1993. (171-178).

[21] Jacas, J., Recasens, J. Fixed points and generators of fuzzy relations. J. Math. Anal. Appl 186 (1994) 21-29.

[22] Jacas, J., Recasens, J. Fuzzy T-transitive relations: eigenvectors and generators. Fuzzy Sets and Systems 72 (1995) 147-154.

[23] Jacas, J., Recasens, J. One dimensional indistinguishability operators. Fuzzy Sets and Systems 109 (2000) 447-451.

[24] Jacas, J., Recasens, J. Maps and Isometries between indistinguishability operators. Soft Computing 6 (2002). 14-20.

[25] Jacas, J, Recasens, J. Aggregation of T-Transitive Relations. Int J. of Intelligent Systems 18 (2003) . 1193-1214.

[26] Jacas, J., Recasens, J. A Calculation of Fuzzy Points. FUZZ'IEEE2006. Vancouver. (2006).

[27] Jacas, J., Recasens, J. The Group of Isometries of an Indistinguishability Operator. Fuzzy Sets and Systems 146 (2004) 27-41.

[28] Jacas, J., Recasens, J. Fuzzy numbers and equality relations. Proceedings FUZZ'IEEE-93 Congress. San Francisco. (1993). 1298-1301.

[29] R. Kruse, J. Gebhardt, F. Klawonn *Foundations of Fuzzy Systems*. Wiley, Chichester (1994)

[30] Klawonn, F. Fuzzy Points, Fuzzy Relations and Fuzzy Functions. In: V. Nov'ak, I. Perfilieva (eds.): Discovering the World with Fuzzy Logic. Physica-Verlag, Heidelberg (2000) 431-453.

[31] R. Lowen, Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. 56 (1976) 623-633.

[32] Menger, K. Untersuchungen über allgemeine Metrik. Math. Ann. 100 (1928) 75-113.

[33] E. P. Klement, R. Mesiar, E. Pap. *Triangular norms*. Kluwer (2000). Dordrecht.

[34] Nehad N. Morsi and M. M. Yakout. Axiomatics for fuzzy rough sets. Fuzzy Sets and Systems 100 (1998) 327-342.

[35] Pawlak, Z. Rough sets. Internat. J. Comput. Inform. Sci. 11 (1982) 341-356.

[36] Pradera, A., Trillas, E., Castiñeira, E. On the aggregation of some classes of fuzzy relations. In L. Magdalena, B. Bouchon-Meunier, J. Gutiérrez-Ríos and R.R. Yager Eds. Technologies for Constructing Intelligent Systems 2: Tools. Studies in Fuzziness and Soft Computing. Springer Verlag 125-136 (2002).

[37] Pradera, A., Trillas, E. A note on Pseudometrics Aggregation. *Int. J. General Systems* **31** 41–51 (2002).

[38] Recasens, J. On a geometric combinatorial problem. Discrete Mathematics 184 (1998) 273-279.

[39] Recasens, J. Indistinguishability Operators. Modelling Fuzzy Equalities and Fuzzy Equivalence Relations Series: Studies in Fuzziness and Soft Computing, Vol. 260. 2011. Springer Verlag. ISBN 978-3-642-16221-3.

[40] Trillas, E., Alsina, C. *Introducción a los espacios métricos generalizados.* Fund. March. Serie Universitaria 49 (1978) Madrid.

[41] Trillas, E. Sobre funciones de negación en la teoría de conjuntos borrosos. Stochastica 3 (1979) 47-59.

[42] Trillas, E. Assaig sobre les relacions d'indistingibilitat. Proc. Primer Congrés Català de Lògica Matemàtica (1982) Barcelona 51-59.

[43] Valverde, L. On the Structure of F-indistinguishability Operators. Fuzzy Sets and Systems 17 (1985) 313-328.

[44] Zadeh. L.A. Similarity relations and fuzzy orderings. Inform. Sci. 3 (1971) 177-200.